Structure and arithmetic of multivariate Ore extensions

André Leroy[a] * Huda Hamdan[b] †

[a]Artois Université, UR 2462 (LML), Faculté of Sciences, 62 300, Lens, France.

[b] Department of Mathematics, Faculty of Science, Damietta University, Damietta 34517, Egypt.

Abstract: We give the basic structure of the multivariable Ore extensions $S = A[\underline{t}; \sigma, \underline{\delta}]$ introduced in the work of Martínez-Peñas and Kschischang. The Pseudo multilinear transformations (PMT's) are introduced and correspond to modules over S. These maps are strongly connected to the evaluation of polynomials in S. A general product formula is obtained. PMT's help to put some structure on the set of roots of a polynomial $f(t) \in S$.

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1 Introduction

The evaluation of polynomials is at the heart of many areas of mathematics. The Ore extension rings (or skew polynomial rings) are one of the most engaging notions of polynomials in noncommutative algebra. The first appearance of Ore extension $K[t; \sigma, \delta]$ dates back to Ore (cf. [14]) in 1933. Numerous authors studied skew polynomials and their evaluations in particular when the coefficient ring is a division ring or a prime ring (cf. e.g. [11]).

*andre.leroy@univ-artois.fr

†huda_merdach@du.edu.eg

Ore extensions have been used in ring theory as a source of examples (cf. e.g. [7], [11]) they also give useful tools in quantum groups [5]. Furthermore, they appeared more recently in coding theory (cf. e.g. [1], [2], [3]).

This paper is concerned with a construction of a noncommutative polynomial ring, denoted $S = A[\underline{t};\sigma,\underline{\delta}]$, that is essentially due to U. Martínez-Peñas and F. R. Kschischang (cf. [13]). The n variables $t_1, t_2, \ldots, t_n$ are free variables and this extension S has a different behavior than the "usual" iterated Ore extension (cf. [10]). We slightly extend the context by considering a general ring A for the coefficients of the polynomials.

In Section 2, some basic properties and examples are given. We introduce the PMT. These maps are our main tool. The use of PMT allows a study of both the left S-modules and their morphisms. This generalizes previous works that appear in case of one variable (cf. [11], [12]). This is given in Proposition (2.6). The PMT's also play a fundamental role in the evaluation of an element $f(\underline{t}) \in S$. We give a lot of examples in (2.3) and (2.5). One of the main results in this section is a complete description of the left $S = A[\underline{t};\sigma,\underline{\delta}]$-modules and their morphisms (see in particular, Proposition (2.7)).

In Section 3, we determine the center of S when the base ring is a division ring, in Proposition (3.1). Also, we introduce the semi-invariant polynomials and construct several examples in (3.4) and (3.6). In Theorem (3.5) we give, under some hypothesis, the structure of semi-invariant polynomials.

In section 4, the evaluation of polynomials is presented. This is completely different from the evaluation in iterated Ore extensions defined in [10]. In addition, we study the relations between evaluation and PMT in Proposition (4.3). In particular, we obtain a general product formula in Proposition (4.4) even when the base ring is not a division ring. We define a relation $\sim$ between elements in A^n. In Proposition (4.5) and Proposition (4.7), PMT's are used to describe the decomposition of the set $V(f) = \{\underline{a} \in A^n \mid f(\underline{a}) = 0\}, f(\underline{t}) \in S = A[\underline{t},\sigma,\underline{\delta}]$, into its $\sim$ classes.

In the last section, we introduce $(\sigma,\underline{\delta})$-centralizer. We give different characterizations of these sets in Proposition (5.2). To each element $\underline{a} \in A^n$ we attach, in Proposition (5.4), a PMT $T_{\underline{a}}$ and show that $T_{\underline{a}}$ is right linear over the $(\sigma,\underline{\delta})$-centralizer of $\underline{a}$. Finally, for a domain A, and an element $f \in S = A[\underline{t};\sigma,\underline{\delta}]$, we describe the set of roots of a polynomial $V(f(\underline{t}))$ in terms of the kernel of $f(T_{\underline{a}})$. The main result for this section is Proposition

(5.5) that presents some structure on the set of roots of polynomial $V(f(\underline{t}))$.

All the rings will be associative with identity.

2 Structure of multivariate Ore extensions

In this section, we introduce our main objects and the tools that we will use. In particular, the Pseudo Multivariate Transformations are defined and applications of these maps are given in (cf. [13]).

Definition 2.1. *Consider a ring A, n variables $t_1, \ldots, t_n$, $\sigma : A \to M_n(A)$ a ring homomorphism, and a sequence of n additive maps $\delta_1, \ldots, \delta_n$. We denote by M the free monoid generated by the variables $\{t_1, \ldots, t_n\}$ and by $S = A[\underline{t}; \sigma, \underline{\delta}]$ the set of polynomials of the form $\sum_{m \in M} \alpha_m m$, where $\alpha_m \in A$ and $m \in M$. On this set, we define the natural addition and we introduce a multiplication based on the concatenation in M and on the following commutation rules:*

$$\forall\, 1 \leq i \leq n,\ \forall\, a \in A, \quad t_i a = \sum_{j=1}^{n} \sigma(a)_{ij} t_j + \delta_i(a). \tag{1}$$

For editorial reasons, for $a \in A$, we will write $\sigma_{ij}(a)$ instead of $\sigma(a)_{ij}$, viewing σ_{ij} as a map from A to A. The next proposition gives some key features of this construction. We leave the proof to the reader.

Proposition 2.2. *(1) The associativity of the ring S leads to the following rule for the maps $\delta_1, \ldots, \delta_n$:*

$$\forall a, b \in A, \quad \delta_i(ab) = \sum_{j=1}^{n} \sigma_{ij}(a)\delta_j(b) + \delta_i(a)b. \tag{2}$$

In a compact form, this can be written as $\underline{\delta}(ab) = \sigma(a)\underline{\delta}(b) + \underline{\delta}(a)b$. The sequence of maps $\delta_{\underline{a}}$ will be called a σ-derivation.

(2) The fact that σ and $\underline{\delta}$ satisfy the above properties can also be summarized by asking that the map ϕ from A to the matrix ring $M_{(n+1)\times(n+1)}(A)$ defined by

$$\phi :\ A \to M_{(n+1)\times(n+1)}(A) \text{ with } a \mapsto \begin{pmatrix} \sigma(a) & \underline{\delta}(a) \\ 0 & a \end{pmatrix},$$

is a ring homomorphism.

Examples 2.3. 1. Let $\underline{a} = (a_1, \ldots, a_n)^t \in A^n$. We define $\delta_{\underline{a}}(x) = \underline{a}x - \sigma(x)\underline{a}$ in other words, $\delta_{\underline{a}} = (\delta_{a_1}, \delta_{a_2}, \ldots, \delta_{a_n})^t$ where $\delta_{a_i}(x) = a_i x - \sum_{j=1}^{n} \sigma_{ij}(x)a_j$. One can check that $\delta_{\underline{a}}$ is indeed a σ-derivation. When $\delta = \delta_{\underline{a}}$, we can erase the derivation in the sense that, $A[\underline{t}; \sigma, \delta_{\underline{a}}] = A[\underline{t} - \underline{a}; \sigma]$.

2. Similarly if there exist $U \in Gl_n(A)$ and $\tau_1, \ldots, \tau_n$ automorphisms of the ring A, such that, for every $x \in A$, we have

$$\sigma(x) = U(diag(\tau_1(x), \ldots \tau_n(x)))U^{-1}$$

then, noting $\tau = diag(\tau_1, \ldots, \tau_n)$ and $\underline{y} = U^{-1}\underline{t}$, we get, for any $x \in A$, $\underline{y}x = U^{-1}\underline{t}x = U^{-1}(\sigma(x)\underline{t} + \underline{\delta}(x)) = U^{-1}\sigma(x)\underline{t} + U^{-1}\underline{\delta}(x) = U^{-1}\sigma(x)UU^{-1}\underline{t} + U^{-1}\underline{\delta}(x) = \tau(x)\underline{y} + U^{-1}\underline{\delta}(x)$. One can check that $U^{-1}\underline{\delta}(x)$ is a τ-derivation, so that we can write

$$A[\underline{t}; \sigma, \underline{\delta}] = A[\underline{y}; \tau, U^{-1}\underline{\delta}].$$

3. Assume that $A = K$ is a division ring finite-dimensional over its center k and that $\sigma(\alpha) = diag(\alpha, \ldots, \alpha) \in M_n(K)$ for any $\alpha \in k$, then by a direct application of the Skolem Noether theorem (cf. Cohn, P. M. Book [8], p. 262) we obtain that there exists an invertible matrix $U \in Gl_n(K)$ such that $\sigma(a) = U diag(a, \ldots, a)U^{-1}$ for every $a \in K$. In particular, using the previous item we get that

$$K[\underline{t}, \sigma, \underline{\delta}] = K[\underline{y}; Id., U^{-1}\underline{\delta}]$$

where $\underline{y} = U^{-1}\underline{t}$.

4. If σ is diagonal, in other words if $\sigma = diag(\sigma_1, \ldots, \sigma_n)$ then, for any $1 \leq i \leq n$, the commutation rules are $t_i a = \sigma_i(a)t_i + \delta_i(a)$, where δ_i is a σ_i-derivation. In this case, the Ore extension $A[\underline{t}; \sigma, \underline{\delta}]$ contains all the one variable Ore extensions $A[t_i; \sigma_i, \delta_i]$.

5. Let A be a ring, $\alpha, \beta \in End(A)$, and γ be an (α, β)-derivation (i.e. $\gamma \in End(A, +)$ and, for any $a, b \in A$ we have $\gamma(ab) = \alpha(a)\gamma(b) + \gamma(a)\beta(b)$). We can check that the map

$$\sigma : A \longrightarrow M_2(A) : \ a \mapsto \begin{pmatrix} \alpha(a) & \gamma(a) \\ 0 & \beta(a) \end{pmatrix}$$

is a homomorphism of rings. If $x \in A$ we can define an (α, β)-derivation γ via $\gamma(a) = x\beta(a) - \alpha(a)x$. Such an (α, β)-derivation is called inner. For more information on (α, β)-derivations the reader may consult [4]. The map σ above gives rise to the extension $A[(t_1, t_2)^t; \sigma]$.

6. Let us notice that in the case of an upper triangular σ of the form

$$\sigma(a) = \begin{pmatrix} \alpha(a) & \delta(a) \\ 0 & a \end{pmatrix}$$

We get that $\delta : A \longrightarrow A$ is an α-derivation and we can consider both $R = A[t; \alpha, \delta]$ and $S = A[\underline{t}; \sigma]$ where $\underline{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$. Let us remark that the map $\varphi : S \to R$ defined by $\varphi(t_1) = t$, $\varphi(t_2) = 1$ and $\varphi(a) = a$ for all $a \in A$ is a ring homomorphism between S and R.

7. We can generalize the points (5) and (6) above as follows. Let A be a ring, $\alpha : A \longrightarrow M_n(A)$, and $\beta : A \longrightarrow M_l(A)$ be morphisms of rings. A map $\gamma : A \longrightarrow M_{n\times l}(A)$ is an (α, β)-derivation if γ is additive and satisfies $\gamma(ab) = \alpha(a)\gamma(b) + \gamma(a)\beta(b)$. As above, this leads to

$$\sigma : A \longrightarrow M_{n\times l}(A) : \ a \mapsto \begin{pmatrix} \alpha(a) & \gamma(a) \\ 0 & \beta(a) \end{pmatrix}$$

and we get a multivariable extension with $n+l$ variables $A[(t_1, \ldots, t_{n+l})^t; \sigma]$. As a special case, we can consider an inner (α, β)-derivation via a matrix $x \in M_{n\times l}(A)$ and define, for $a \in A$, $\gamma(a) = x\beta(a) - \alpha(a)x$. We leave to the reader the analogue of (6).

We now introduce the important notion of PMT. We keep our usual notation $S = A[\underline{t}; \sigma, \underline{\delta}]$. If V is a left S-module, then V is also a left A-module and, for any $1 \leq i \leq n$, the action of t_i on V must satisfy the following equality

$$(t_i a).v = (\sum_j \sigma_{ij}(a) t_j + \delta_i(a)).v. \qquad (3)$$

This leads to the next definition.

Definition 2.4. *Let V be a left A-module and $T_1, \ldots, T_n \in \mathrm{End}(V, +)$ be such that, for $a \in A$ and $v \in V$, we have*

$$\forall\, 1 \leq i \leq n, T_i(a.v) = \sum_{j=1}^{n} \sigma_{ij}(a) T_j(v) + \delta_i(a).v. \tag{4}$$

A sequence of maps satisfying these equations will be called a $(\sigma, \underline{\delta})$-pseudo-multilinear transformation ($(\sigma, \underline{\delta})$-PMT, for short) on V.

In other words, writing $T = (T_1, T_2, \ldots, T_n)^t$ for a column of elements in $\mathrm{End}(V, +)$, we can write the equality in (cf. equation 4) in a compact form as follows:

$$T(a.v) = \sigma(a)T(v) + \underline{\delta}(a)v.$$

Examples 2.5. (a) One can check that the sequence $\underline{\delta} = (\delta_1, \ldots, \delta_n)^t$ is a PMT on A.

(b) Let $\underline{a} = (a_1, \ldots, a_n)^t$ be a column $\in A^n$ then the PMT on A defined as follows $T_{\underline{a}} = (T_{a_1}, \ldots, T_{a_n})^t$ with

$$T_{a_i}(b) = \sum_{j=1}^{n} \sigma_{ij}(b) a_j + \delta_i(b). \tag{5}$$

We can check that we indeed get a PMT defined over A. As we will see, this PMT is closely related to the evaluation at $\underline{a}$.

(c) Let us remark that if we consider $\underline{a} = (0, \ldots, 0)^t \in A^n$, then the PMT $T_{\underline{a}}$ is simply the map $(\delta_1, \ldots, \delta_n)^t$.

As in the case of a single variable, we can associate a ring homomorphism to each PMT. This is the purpose of the next proposition.

Proposition 2.6. *Let T be a PMT defined on left S-module V. Then*

(1) The following map

$$\varphi : S \to \mathrm{End}(V, +) \quad \textit{such that} \quad \varphi(f(\underline{t})) = f(\underline{T}),$$

is a ring homomorphism.

(2) There is a 1-1 correspondence between the set of PMT's and the set of S-modules.

Proof. (1) The map φ is additive and we only need to check that it is also multiplicative. We have, for every $a \in A$ and $1 \le i \le n$, $T_i L_a = \varphi(t_i a) = \varphi(\sum_j \sigma_{ij}(a) t_j + \delta_i(a)) = \sum_j \sigma_{ij}(a) T_j + L_{\delta_i(a)}$.

(2) If $T = (T_1, \ldots, T_n)$ is a PMT on a module ${}_A V$ we obtain a left $S = A[\underline{t}, \sigma, \underline{\delta}]$-module structure on V by defining $t_i.v = T_i(v)$. On the other hand when ${}_S V$ is a left S-module the actions of $t_1, \ldots, t_n$ on V give a PMT on V as in the paragraph before the definition 2.4. □

If ${}_S V$ is a left S-module such that ${}_A V$ is free of dimension l and if B is a basis of V, the actions of $t_1, \ldots, t_n$ on V are completely described by n matrices $\{\tau_1, \ldots, \tau_n\} \subset M_l(A)$ expressing these action on the basis. These matrices are sufficient to describe the left S-module structure of V. Suppose that V_1 and V_2 are two left $S = A[\underline{t}; \sigma, \underline{\delta}]$-modules such that both ${}_A V_1$ and ${}_A V_2$ are free with basis $\beta_1 = \{e_1, \ldots, e_{n_1}\}$ and $\beta_2 = \{u_1, \ldots, u_{n_2}\}$ respectively. We denote the matrices corresponding to these actions in the respective basis by $X_1, \ldots, X_n \in M_{n_1 \times n_1}(A)$ and $Y_1, \ldots, Y_n \in M_{n_2 \times n_2}(A)$. If $V_1 \xrightarrow{\varphi} V_2$ is a left A-morphism, we let $M \in M_{n_2 \times n_1}(A)$ to be the matrix representing φ in the basis B_1 and B_2.

Now, suppose that $S = A[\underline{t}; \sigma, \underline{\delta}]$ be a multivariate Ore extension. For $i = 1, 2$, $T_i = (T_{i1}, \ldots, T_{in})^t$ be (σ, δ)-PMT defined on V_1 and V_2, respectively. If $\varphi \in Hom_A(V_1, V_2)$ is an A-module homomorphism, also $M \in M_{n_1 \times n_2}(A)$, $X = (X_1, \ldots, X_n) \in M_{n_1 \times n_1}(A)$ and $Y = (Y_1, \ldots, Y_n) \in M_{n_2 \times n_2}(A)$ denote matrices representing φ, T_1 and T_2 respectively in the appropriate basis β_1 and β_2. Let ${}_S V_1$ and ${}_S V_2$ be the left R-module structures corresponding to T_1 and T_2, respectively. We have the following properties:

Theorem 2.7. *The following conditions are equivalent:*

(i) $\varphi \in Hom_S(V_1, V_2)$;

(ii) $\varphi T_{1i} = T_{2i} \varphi$, *for every* $1 \le i \le n$;

(iii) $X_iM = \sum_j \sigma_{ij}(M)Y_j + \delta_i(M)$ *for every* $1 \leq i \leq n$.

Proof. Firstly, we have

$$\begin{aligned}
(X_iM)_{l,k} &= (\sum_{js} \sigma_{is}(M_{jk})T_{is}(w_j) + \delta_i(M_{jk})w_j)_l \\
&= (\sum_{j,s} \sigma((M_{ik})_{js}) \sum_{p=1}^{n_2} (Y_s)_{pj} w_p + \delta_i(M_{jk})w_j)_l \\
&= \sum_{j,s} \sigma((M_{jk})_{is})(Y_s)_{lj} + \delta_i(M_{lk}) \\
&= \sum_s (\sum_j \sigma_{is}(M_k)(Y_s)_{lj}) + \delta_i(M_{lk}) \\
&= ((\sum_s \sigma_{is}(M)Y_s) + \delta_i(M))_{lk} = \sum_s (\sum_j \sigma_{is}(M_{lj})(Y_s)_{jk})
\end{aligned}$$

Also, $(\varphi \circ T_{i1})_{lk} = (\varphi(T_{i1}(v_k)))_l = \varphi(\sum_{j=1}^{n_2}(X_i)_{jk}v_j)_l = (\sum_{j=1}^{n_2}(X_i)_{jk}\varphi(v_j))_l = (\sum_{j=1}^{n_2}(X_i)_{jk}(\sum_s M_{sj}w_s))_l = \sum_{j=1}^{n_2}(X_i)_{jk}M_{lj}$.
Now (i) $\Leftrightarrow$ (ii) $\varphi(t_i.v_j) = t_i.\varphi(v_j) \Leftrightarrow \varphi(T_{1i}(v_j)) = T_{i2}(\varphi(v_j)) \Leftrightarrow (\varphi \circ T_{1i}(v_j) = (T_{i2} \circ \varphi)(v_j)$.
(ii)$\Leftrightarrow$ (iii) $M(\varphi \circ T_{1i})_{lk} = (\varphi \circ T_{1i})(v_l)_k = (M(T_{1i})M(\varphi))_{lk} = (X_iM)_{lk}$

On the other hand,

$$\begin{aligned}
M(\varphi \circ T_{1i})_{lk} = M(T_{2i} \circ \varphi)_{lk} &= \sum_k ((T_{2i} \circ \varphi)(v_l))_k) w_k \\
&= \sum_k (T_{2i}(\varphi(v_l)))_k w_k = (T_{2i}(\sum_j M_{lj}w_j))_k \\
&= (\sum_j \sum_s \sigma_{is}(M_{lj})T_{2i}(w_j) + \delta_i(M_{lj})w_j)_k \\
&= (\sum_j \sum_s \sigma_{is}(M_{lj}) \sum_m ((Y_i))_{jm}(w_m) + \delta_i(M_{lj})w_j)_k \\
&= (\sum_j \sum_s \sum_m \sigma_{is}(M_{lj})(Y_i)_{jm}w_m + \sum_j \delta_i(M_{lj})w_j)_k \\
&= \sum_{j,s} \sigma_{is}(M_{lj})(Y_i)_{jk} + \delta_i(M_{lk}) = \sum_{j,s} \sigma_{is}(M)_{lj}(Y_i)_{jk} + \delta_i(M_{lk}) \\
&= \sum_s (\sigma_{is}(M)Y_i)_{lk} + \delta_i(M)_{lk} = (\sum_s \sigma_{is}(M)Y_i + \delta_i(M))_{lk}.
\end{aligned}$$

□

A classical feature of one variable Ore extensions is the fact that $R = K[t;\sigma,\delta]$ is embeddable in a division ring when K is itself a division ring. Since R is a left principal domain, this is immediate. Although in our more general setting $S = K[\underline{t},\sigma,\underline{\delta}]$ is not even Noetherian, it is also embeddable in a division ring. We will not use the following theorem and hence mention it with a sketch of proof.

Theorem 2.8. *Let K be a division ring and $S = K[\underline{t},\sigma,\underline{\delta}]$. Then S is embeddable in a division ring.*

Proof. We first show that the ring S is filtered via the length of monomials. Moreover, this filtration satisfies the weak algorithm and hence is a fir (cf. Section 2.4, in particular Theorem 2.4.4 and Theorem 2.4.6 in [6]). We conclude that S is indeed embeddable in a division ring (cf. Corollary 7.5.14 in [6]). □

3 Center of S and Semi-invariant polynomials

The purpose of the next proposition is to study the center of S so, we consider K a division ring and $S = K[\underline{t},\sigma,\underline{\delta}]$ where $\underline{t} = (t_1,...,t_n), \sigma = (\sigma_{ij})$ and $n > 1$. then,

Proposition 3.1. *The center Z(S) of S is*

$$Z(K)_{\sigma,\underline{\delta}} = \{a \in K \mid ab = ba \ \forall\ b \in K; \sigma(a) = a.I_n, \delta_i(a) = 0,\ \forall\, 1 \leq i \leq n\}$$

Proof. Let $P(t) = \sum_{\omega\in\Omega} a_\omega, \omega \in Z(S)$, where Ω is the semigroup generated by $t_1,...,t_n$. We order Ω by the deg lex order with $t_1 < ... < t_n$
Let $\alpha\omega$ be the leading term of $P(t)$ ($\alpha \in K^*, \omega \in \Omega$). Since $P(t)$ is central and deg lex is a term order. $\forall\ i \in \{1,...,n\}$

$$t_i\alpha\omega = \alpha\omega t_i, \quad \forall\ 1 \leq i \leq n$$

$$\sum_j \delta_{ij}(\alpha)t_j\omega + \delta_i(\alpha)\omega = \alpha\omega t_i$$

Comparing leading terms we have

$$\delta_{in}(\alpha)t_n\omega = \alpha\omega t_i \quad \forall\ 1 \leq i \leq n$$

$$\alpha^{-1}\sigma_{in}(\alpha)t_n\omega = \alpha\omega t_i$$

From this we conclude $\omega = 1$

$$\alpha^{-1}\sigma_{in}(\alpha)t_n = t_i \qquad \forall\, 1 \le i \le n$$

$;\sigma_{in}(\alpha) = 0,\ \forall\ \sigma \in \{1, ..., n-1\}$ *and* $\sigma_{nn}(\alpha) = \alpha$
Now $P(t) \in Z(S) \implies P(t) \in K$ So, $P(t) = \alpha \in Z(S)$, $\forall\ i$ we have $t_i\alpha = \alpha t_i \implies \sum_j \sigma_{ij}(\alpha)t_j + \delta_i(\alpha) = \alpha t_i$

$\forall\ i, j \in \{1, ..., n\}, \sigma_{ij}(\alpha) = 0\ if\ i \neq j\ ;\ \sigma(\alpha) = \begin{pmatrix} \alpha & ... & 0 \\ & \ddots & \\ 0 & ... & \alpha \end{pmatrix}, \sigma_{ii}(\alpha) = \alpha$

Moreover, $\delta_i(\alpha) = 0\ \ \forall\ i$ and $\alpha a = a\alpha \quad \forall\ a \in K \implies \alpha \in Z(K)_{\sigma,\underline{\delta}}$. □

Definition 3.2. *A nonzero polynomial* $p(\underline{t}) \in S$ *is right semi-invariant if for any* $a \in K$ *there exists an* a' *in* K *such that* $p(\underline{t})a = a'p(\underline{t})$.

Lemma 3.3. *Suppose that* $p(\underline{t}) \in S$ *is right semi-invariant. Then there exists a homomorphism* φ *from* K *to* K *such that* $p(\underline{t})a = \varphi(a)p(\underline{t})$.

Proof. Let us notice that for $a \in K$, there exists a unique element $a' \in K$ such that $p(t)a = a'p(t)$. Since the element a' is unique we can define the map $\varphi : K \to K$ such that $\varphi(a) = a'$. It is easy to check that φ is a ring homomorphism. □

Examples 3.4. (1) Let K be a division ring, and consider a map $\sigma = diag(\sigma_1, \sigma_2)$ and $\delta = (\delta_1, \delta_2) = (0, 0)$. Assume that $\sigma_1^l = \sigma_2^l$, then one can check that $t_1^l + t_2^l$ is a semi-invariant polynomial in $S = K[\underline{t}, \sigma, \underline{\delta}]$.

(2) Let K be a division ring of characteristic 2, and consider maps $\sigma = diag(Id, Id)$ and $\delta = (\delta_1, \delta_2)$, where be two usual derivations on K are such that $\delta_1^2 = \delta_2^2 = 0$, then $t_1^2 + t_2^2$ is a semi-invariant polynomial in $S = K[\underline{t}, \sigma, \underline{\delta}]$.

Let us remark that a' is unique. In the case when $n = 1$ these semi-invariant polynomials are at the heart of the structure theory since such a nonconstant semi-invariant polynomial exists if and only if the Ore extension is not simple. In our general frame, the semi-invariant notion is too rigid to give any structure result. Nevertheless in some particular cases, these polynomials exist and their zeroes behave nicely. We will analyze this behavior in the next section and now we will just construct these polynomials. In the case when σ is diagonal, say $\sigma = diag(\sigma_1, \ldots, \sigma_n)$ we can search the semi-invariant polynomials in the subrings $K[t_i, \sigma_i, \delta_i]$, where $1 \leq i \leq n$.

Theorem 3.5. *Let $S = K[\underline{t}, \sigma, \underline{\delta}]$ be a multivariate Ore extension such that there exists $1 \leq i \leq n$ with $\sigma = diag(\sigma_1, \ldots, \sigma_n)$ where $\sigma_i \in Aut(K)$. Then the skew polynomial $S_i = K[t_i, \sigma_i, \delta_i]$ is contained in S. We assume that there exists a nonconstant semi-invariant polynomial $p_i(t_i) \in S_i$. Then*

1. *For $1 \leq i \leq n$, the ring S_i is not simple if and only if there exists a monic nonconstant semi-invariant polynomial of minimal non zero degree, say $p_i(t_i) \in S_i$.*

2. *Suppose that $p_i(t_i)$ is as in (1) then all the monic semi-invariant polynomials contained in S_i are of the form $\sum_{j=0}^{l} a_j p_i(t_i)^j$ for some $a_j \in K$ with $a_l = 1$.*

Proof. These results are extracted from (cf. [9]). □

Examples 3.6.

1. if $\sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\delta_i = 0$ then, for any $a \in A$, $t_i a = \sigma_i(a) t_i$. This shows that t_i is semi-invariant (even invariant).

2. If there exists $1 \leq i \leq n$ such that for every $1 \leq j \leq n$ we have $\sigma_{ij} = \sigma_i \delta_{ij}$ (where δ_{ij} stands for the classical Kroeneker symbol) and δ_i is quasi algebraic (cf. [9]) then there exists a monic invariant polynomial $p(t_i)$, say of degree l, such that $p(\delta_i)(x) = \sigma_i^l(x) p(t_i)$ so that the polynomial $p(t_i)$ is semi-invariant.

3. Let α, β, γ be as in Examples (cf. 2.3) part number (5)). Suppose that $\alpha\gamma = -\gamma\beta$ then one can check that t_1^2 and t_2^2 are semi-invariant polynomials in $A[(t_1, t_2)^t, \sigma]$.

4. Let us now give an example of a multivariate Ore extension S that is simple. This will be very similar to the Weyl algebra construction. We start with the field of rational fractions $k(x)$ over a field k of characteristic zero and define $\sigma : k(x) \to M_2(k(x))$ and $\delta_1 = \delta_2$ via

$$\sigma(p(x)) = \begin{pmatrix} p(x) & 0 \\ 0 & p(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \delta_1(p(x)) \\ \delta_2(p(x)) \end{pmatrix} = \begin{pmatrix} p'(x) \\ p'(x) \end{pmatrix}$$

We will show that $S = k(x)[\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \sigma, \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}]$ is simple. define the usual deglex order on the monomials in the variables t_1, t_2. Assume that I is a nonzero two-sided ideal of S and let $f = f(t_1, t_2) \in I$ be nonzero polynomial with minimal deglex order amongst nonzero elements of I. If $f \in k(X)$ we get that f is invertible and hence $I = S$. So let $w \neq 1$ be the deglex leading monomial in f. An easy computation shows that the deglex order of $xw - wx$ is smaller than that of w. Hence the the deglex order of $fx - fx \in I$ is smaller than that of f. This implies that $fx = xf$, and hence the same is true for the leading term of f. This implies that $f \in k(x)$ a contradiction.

5. Let us notice that in the previous example when the characteristic of k is finite, the ring S will not be simple anymore. For instance if $char(k) = 2$, we have that the left ideal generated by $I = St_1^2 + St_2^2 + \sum_{w \in \Omega} St_1^2 t_2 w + \sum_{w \in \omega} t_2^2 t_1 w$ is a two sided ideal of S. It is easy to check that $\delta_1^2 = \delta_2^2 = 0$ and this implies that the elements t_1^2 and t_2^2 are in the kernel of the ring homomorphism (see Proposition (2.6)) $\varphi : S \to End(k(x), +)$ which is associated to the MLT defined by the point $(0, 0)$.

4 Evaluation and $(\sigma, \underline{\delta})$-conjugation

The evaluation of polynomials is a classical subject of study. We define the evaluation of a polynomial $f(\underline{t}) \in K[\underline{t}, \sigma, \underline{\delta}]$ at an element $\underline{a} \in K^n$.

Definitions 4.1. *1. We define the evaluation of $f(\underline{t}) \in S = A[\underline{t}; \sigma, \underline{\delta}]$ at $(a_1, \ldots, a_n) \in A^n$, via the representative of $f(\underline{t}) + I \in S/I$ by an element of A, where I is the left ideal $I = S(t_1 - a_1) + S(t_2 - a_2) + \cdots + S(t_n - a_n)$.*

2. If $x \in U(A)$ we denote $\underline{a}^x$ the $(\sigma, \underline{\delta})$-conjugate of $\underline{a}$ (a column in A^n) by x defined by

$$\underline{a}^x = \sigma(x)\underline{a}x^{-1} + \underline{\delta}(x)x^{-1} \tag{6}$$

3. For $\underline{a}, \underline{b} \in A^n$ we define $\underline{a} \sim \underline{b}$ if there exists a nonzero divisor $x \in A$ such that $\underline{b}x = \sigma(x)\underline{a} + \underline{\delta}(x)$. We put

$$\Delta(\underline{a}) = \{\underline{b} \in A^n \mid \underline{a} \sim \underline{b}\}. \tag{7}$$

It is important to remark that for a general ring A, the relation in (cf. equation (7)) is not symmetric and hence doesn't lead to an equivalence relation.

Examples 4.2. 1. If we suppose $n = 2$, then evaluating t_1t_2 at (a_1, a_2) we get $(t_1t_2)(a_1, a_2) = \sigma_{11}(a_2)a_1 + \sigma_{12}(a_2)a_2 + \delta_1(a_2)$. Let us now compare with t_2t_1 evaluated at (a_1, a_2). We also have $(t_2t_1)(a_1, a_2) = \sigma_{22}(a_1)a_2 + \sigma_{21}(a_1)a_1 + \delta_2(a_1)$.

2. When $\sigma = (\sigma_1, \dots, \sigma_n)$ is diagonal we have, for $1 \leq i \leq n$ and $a \in A$, $t_ia = \sigma_i(a) + \delta_i(a)$ and hence, the skew polynomial rings $A[t_i, \sigma_i, \delta_i]$ are contained in S. We compute $(t_1t_2)(a_1, a_2) = \sigma_1(a_2)a_1 + \delta_1(a_2)$ and $(t_2t_1)(a_1, a_2) = \sigma_2(a_1)a_2 + \delta_2(a_1)$.

Let us remark that the evaluations that we obtain in the above examples are very different from the evaluations that appear when considering iterated extensions (cf. [10]).

Since S/I is a left S-module, it gives rise to a $(\sigma, \underline{\delta})$-PMT on S/I given by the actions of t_i for $1 \leq i \leq n$. The elements of S/I are represented by a unique element of A so that the action of t_i on S/I can be described by

$$t_i.(x + I) = t_ix + I = \sum \sigma_{ij}(x)a_j + \delta_i(x) + I.$$

The PMT attached to this action is $T_{\underline{a}} = (T_{a_1}, T_{a_2}, \dots, T_{a_n})$ where, for $x \in A$ and $1 \leq i \leq n$, we have $T_{a_i}(x) = \sum_{j=1}^{n} \sigma_{ij}(x)a_j + \delta_i(x)$ (cf. Examples 2.5 equation number (5)).

Let us recall from Proposition (2.6) that the map $\varphi_{\underline{a}} : S = A[\underline{t}, \sigma, \underline{\delta}] \to End(A, +)$ defined by $\varphi_{\underline{a}}(f(t_1, \dots, t_n)) = f(T_{a_1}, \dots, T_{a_n})$ is a ring homomorphism.

The link between evaluation and PMT is given in the next Proposition.

Proposition 4.3. *For* $f(\underline{t}) \in S = A[\underline{t}, \sigma, \underline{\delta}]$ *and* $\underline{a} \in A^n$ *we have*

$$f(\underline{a}) = f(T_{\underline{a}})(1).$$

Proof. Since $f(\underline{t})$ is a sum of monomials, it is enough to prove this formula for a monomial. Let $w = t_{i_1}t_{i_2}\cdots t_{i_l}$ be such a monomial. We proceed by induction on the length of w. If this length is one, we have $w = t_{i_1}$ for some $1 \leq i_1 \leq n$. Since $\sigma(1) = I_n$, we have that $T_{a_i}(1) = a_{i_1}$. Hence, $t_{i_1}(\underline{a}) = a_{i_1} = T_{a_i}(1)$.

Assume that the formula is true for monomials of length l, for some $l \geq 1$, and consider a monomial of length $l+1$: $w = w't_i$ where w' is of length l. We then have $w(\underline{a}) = (w't_i)(\underline{a}) = (w'(t_i - a_i) + w'a_i)(\underline{a})$ and since $w'(t_i - a_i) \in S(t_i - a_i)$ we have $w(\underline{a}) = (w'a_i)(\underline{a})$. Using the induction hypothesis we obtain $w(\underline{a}) = (w'a_i)(T_{\underline{a}})(1) = (\varphi_{\underline{a}}(w'a_i))(1) = (\varphi_{\underline{a}}(w') \circ \varphi_{\underline{a}}(a_i))(1) = w'(T_{\underline{a}})(a_i) = w'(T_{\underline{a}})((T_{a_i})(1)) = w'(T_{\underline{a}})(t_i(T_{\underline{a}})(1)) = (w't_i)(T_{\underline{a}})(1) = w(T_{\underline{a}})(1)$. □

The fact that the map φ in Proposition (2.7) is a ring homomorphism, then immediately leads to part (1) of the following proposition. This formula is called the "product formula".

Proposition 4.4. *Suppose that $f, g \in S$, $\underline{a} \in A^n$, and $x \in A$.*

1. *We have:*

$$(fg)(\underline{a}) = f(T_{\underline{a}})(g(\underline{a}))$$

   In particular, if $g(\underline{t}) = x \in A$, then we have $(f \circ x)(\underline{a}) = f(T_{\underline{a}})(x)$.

2. *Assume that $0 \neq g(a) \in U(A)$, then we get:*

$$(fg)(\underline{a}) = f(\underline{a}^{g(\underline{a})})g(\underline{a}).$$

Proof. (1) $fg(\underline{a}) = fg(T_{\underline{a}})(1) = (\varphi(fg))(1) = (\varphi(f)\circ\varphi(g))(1) = \varphi(f)(\varphi(g)(1))$ $f(T_{\underline{a}})(g(T_{\underline{a}})(1)) = f(T_{\underline{a}})(g(\underline{a}))$, where φ is the map associated to the PMT $T_{\underline{a}}$, as defined in Proposition (2.6).

(2) We put $x = g(\underline{a})$ and $I = \sum S(t_i - a_i^x)$. We have that, $f - f(\underline{a}^x) \in \sum S(t_i - a_i^x)$. Since $(\underline{t} - \underline{a}^x)x = \sigma(x)(\underline{t} - \underline{a})$, we get that $fx - f(\underline{a}^x)x \in \sum S(t_i - a_i^x)x \in \sum S(t_i - a_i)$. This shows that $(fx)(\underline{a}) = f(\underline{a}^x)x$ and hence $(fg)(\underline{a}) = f(T_{\underline{a}})(g(\underline{a})) = f(T_{\underline{a}})(x) = (fx)(\underline{a}) = f(\underline{a}^x)x$. □

The first equality in the previous proposition shows how the use of $T_{\underline{a}}$ leads to a general product formula for polynomials with coefficients in a general base ring A. This also gives a link between the kernel of $f(T_{\underline{a}})$ and the roots of $f(\underline{t})$. In case $A = K$ is a division ring, and $f.x \neq 0$, the fact that $f(T_{\underline{a}})(x) = (f.x)(\underline{a}) = f(\underline{a}^x)x$ shows that the kernel of $f(T_{\underline{a}})$ corresponds to roots of $f(\underline{t})$. The same is true for a domain but requires some formalism.

Proposition 4.5. *Let $\underline{a}, \underline{b} \in A^n$ be such that there exists a nonzero divisor $x \in A$ with $\underline{b}x = \sigma(x)\underline{a} + \underline{\delta}(x)$. Then*

1. *For any $y \in A$, $f(T_{\underline{b}})(y)x = f(T_{\underline{a}})(yx)$*
2. *We have $x \in \ker f(T_{\underline{a}})$ if and only if $f(\underline{b}) = 0$. In particular, if $x \in U(A)$, $x \in ker(f(T_{\underline{a}}))$if and only if $f(\underline{a}^x) = 0$.*

Proof. (1) Let us first compute $T_{\underline{b}}(y)x = \sigma(y)\underline{b}x + \underline{\delta}(y)x = \sigma(y)\sigma(x)\underline{a} + \sigma(y)\underline{\delta}(x) + \underline{\delta}(y)x = \sigma(yx)\underline{a} + \underline{\delta}(yx) = T_{\underline{a}}(yx)$. We use an induction on the length l of a word $w = w(t_1, \ldots, t_n)$ to prove the formula for monomials. If $l = 1$, $w(t_1, \ldots, t_n) = t_i$ for some $1 \leq i \leq n$ and the desired equality is just the i^{th} row of the formula $T_{\underline{b}}(y)x = T_{\underline{a}}(yx)$, that we just proved. Now assume the formula has been proved for a word $w = w(t_1, \ldots, t_n)$ and let us show it holds for $w(t_1, \ldots, t_n)t_i$ where $1 \leq i \leq n$. We have $(w(t_1, \ldots, t_n)t_i)(T_{\underline{b}})(y)x = w(T_{b_1}, \ldots, T_{b_n})(T_{b_i}(y))x$. Thanks to the induction hypothesis we obtain that $w(T_{b_1}, \ldots, T_{b_n})(T_{b_i}(y))x = w(T_{a_1}, \ldots, T_{a_n})((T_{b_i})(y)x)$. Using the formula obtained for $l = 1$, leads to $w(T_{a_1}, \ldots, T_{a_n})((T_{b_i})(y)x) = w(T_{a_1}, \ldots, T_{a_n})(T_{a_i})(yx)$ and we conclude that $(w(t_1, \ldots, t_n)t_i)(T_{\underline{b}})(y)x = w(T_{a_1}, \ldots, T_{a_n})(T_{a_i})(yx)$, as desired. The fact that the formula is true for a polynomial is now obvious.

(2) Considering the equation in (1) with $y = 1$, we get $f(\underline{b})x = f(T_{\underline{b}}(1))x = f(T_{\underline{a}})(x)$ and the fact that x is not a zero divisor immediately gives that $x \in \ker f(T_{\underline{a}})$ if and only if $f(\underline{b}) = 0$. The last assertion is clear. □

We first give a consequence of Proposition (4.4) on the roots of a semi-invariant polynomial. We write $V(f) = \{\underline{a} \in K^n; f(\underline{a}) = 0\} \subset K^n$ for the set of roots of $f \in S$.

Theorem 4.6. *Let $p(\underline{t}) \in S = K[\underline{t}; \sigma, \underline{\delta}]$, where K is a division ring be a semi-invariant polynomial. Then for any $\underline{b} \in V(p)$ we have that $\Delta^{\sigma,\delta}(\underline{b}) \subset V(p)$.*

Proof. By hypothesis, for every $a \in K \setminus \{0\}$, we have that $p(\underline{t})a = \varphi(a)p(\underline{t})$. Hence we have $(p(\underline{t})a)(\underline{b}) = p(\underline{t})(\underline{b}^a)a = (\varphi(a)p(\underline{t}))(\underline{b}) = \varphi(a)p(\underline{b}) = 0$, by the product formula. This shows that for any $a \in K \setminus \{0\}$, we have that $p(\underline{t})(\underline{b}^a) = 0$, as required. □

Proposition 4.7. *Suppose that $\sigma = diag(\sigma_1, \ldots, \sigma_n)$ and that, for some $1 \leq i \leq n$ there is no polynomial $q(t_i) \in S_i = K[t_i, \sigma_i, \delta_i]$ such that $q(a) = 0$*

for every $a \in K$. Then $p_i(t_i) \in S_i$ is semi-invariant if and only if, for every $a \in K$,

$$p_i(T_a) = r_{p_i(T_a)(1)} \circ \sigma_i^{n_i}$$

where $n_i = deg(p_i(t_i))$ and $r_{p_i(T_a)(1)}$ stands for the right multiplication by $p_i(T_a)(1)$.

Proof. Using Proposition(4.4), we have, for any $x \in K$, $p_i(T_a)(x) = (p_i(t_i)x)(a) = (\sigma_i^{n_i}(x)p_i(t_i))(a) = \sigma_i^{n_i}(x)(p_i(T_a)(1)) = r_{p_i(T_a)(1)} \circ \sigma_i^{n_i}(x)$. This gives the formula.

Conversely, if $p_i(T_a) = r_{p_i(T_a)(1)} \circ \sigma_i^{n_i}$ then, for every $x \in K$, $p_i(T_a)(x) = \sigma_i^{n_i}(x)p_i(T_a)(1)$. Thus for every $x, a \in K$, $(p_i(t_i)x)(a) = (\sigma_i^{n_i}(x)p_i(t_i))(a)$. Hence our hypothesis shows that, for any $x \in K$, $p_i(t_i)x = \sigma_i^{n_i}(x)p_i(t_i)$, showing that $p_i(t_i)$ is semi-invariant. □

5 Centralizers and roots

In this section we study the important role played by the centralizer, Also, we will assume A is a (noncommutative) domain and S will stand for $S = A[\underline{t}; \sigma, \underline{\delta}]$.

Definitions 5.1. *Let $\underline{a} = (a_1, \dots, a_n)^t \in A^n$, the $\sigma, \underline{\delta}$ centralizer of $\underline{a}$, denoted $C^{\sigma,\delta}(\underline{a})$ is the set*

$$C^{\sigma,\underline{\delta}}(\underline{a}) = \{x \in A \mid \sigma(x)\underline{a} + \underline{\delta}(x) = \underline{a}x\} \subset A \tag{8}$$

The idealizer, denoted idl(I) *of a left ideal I of $S = A[\underline{t}, \sigma, \underline{\delta}]$ is defined by* idl(I) = {p ∈ S | Ip ⊂ I}.

One can easily check that $C^{\sigma,\underline{\delta}}(\underline{a})$ and idl(I) are in fact subrings of A and S respectively. Moreover $I \subseteq$ idl(I). If we assume that $n = 1$, one can readily check that $T_{\underline{a}}$ is a right linear map over the subring given by $C^{\sigma,\underline{\delta}}(\underline{a}) := \{x \in A \mid T_{\underline{a}}(x) = \underline{a}x\}$. In the case when $A = K$ is a division ring, $C^{\sigma,\underline{\delta}}(a)$ is a division ring isomorphic to $\mathrm{End}_S(S/I)$, where $I = \sum_i S(t_i - a_i)$.

Proposition 5.2. *(1) $b \in C^{\sigma,\delta}(\underline{a})$ if and only if for any $1 \leq i \leq n$ we have $\sum_{j=1}^n \sigma_{ij}(b)a_j + \delta_i(b) - a_i b = 0$.*

(2) If $I = \sum S(t_i - a_i)$, then S/I is a left S module and a right $C^{\sigma,\delta}(\underline{a})$.

(3) There is a ring isomorphism between $C^{\sigma,\delta}(\underline{a})$ and $End_S(S/I)$, where $S = A[\underline{t};\sigma,\underline{\delta}]$ and $I = \sum_{i=1}^n S(t_i - a_i)$. If the base ring A is a division ring these rings are in fact division rings.

(4) We have isomorphisms of rings

$$C^{\sigma,\delta}(\underline{a}) \cong End_S(S/I) \cong \mathrm{idl(I)/I}.$$

Proof. (1) This is a direct consequence of the definition (cf. equation (8)).

(2) The fact that S/I is a right $C^{\sigma,\delta}(\underline{a})$-module is clear since for any $1 \le i \le n$ and any $b \in C^{\sigma,\delta}(\underline{a})$, we have $(t_i - a_i)b = t_i b - a_i b = \sum_{j=1}^n \sigma_{ij}(b)t_j + \delta_i(b) - a_i b = \sum_{j=1}^n \sigma_{ij}(b)(t_j - a_j) + \sum_{j=1}^n \sigma_{ij}(b)a_j + \delta_i(b) - a_i b$. Hence, by (1) we get $(t_i - a_i)b = \sum_{j=1}^n \sigma_{ij}(b)a_j \in I$.

(3) For $b \in C^{\sigma,\delta}(\underline{a})$, we define a map $\psi(b) : S/I \to S/I$ by $\psi(b)(f(t)+I) = f(t)b + I$. This map is well defined since, for any $1 \le i \le n$ and any $s \in S$, we have (in S/I) $\psi(b)(s(t_i - a_i)) = s(t_i - a_i)b = s(\sum_{j=1}^n \sigma_{ij}(b)(t_j - a_j)$, where the last equality is obtained as in (2) above. The map $\psi(b)$ is easily seen to be left S-linear. The fact that ψ is an isomorphism of rings is easy to check. In case A is a division ring, one can check that if $b \in C^{\sigma,\delta}(\underline{a})$ then $b^{-1} \in C^{\sigma,\delta}(\underline{a})$.

(4) The first isomorphism is given in (3) and the second is easy and well-known. □

Remark 5.3. There is a more general point of view: Having a left S module. We put $C = End_S(V)$. We then obtain a (S, C) bimodule structure on V. If we fix $\underline{a} \in A^n$, and consider $V = S/I$ where $I = \sum S(t_i - a_i)$, we get a $(S, C(\underline{a}))$ bimodule structure on S/I. This shed some light on the fact that $T_{\underline{a}}$ is a $C(\underline{a})$ is a right module map.

Proposition 5.4. *Let $\underline{a} \in A^n$ then for any $1 \le i \le n$, we have*

$$T_{a_i} \in End(A_C), \text{ where } C = C^{\sigma,\delta}(\underline{a}).$$

Proof. We have, for $x \in A$ and $y \in C$, $T_{a_i}(xy) = \sum_j(\sigma_{ij}(xy)a_j + \delta_i(xy)) = \sum_j(\sigma_{ij}(x)\sigma_{ij}(y)a_j + \sigma_{ij}(x)\delta_j(y) + \delta_i(x)y) = \sum_j(\sigma_{ij}(x)(\sigma_{ij}(y)a_j + \delta_j(y)) + \delta_i(x)y) = \sum_j(\sigma_{ij}(x)a_i y + \delta_i(x)y) = \sum_j T_{a_i}(x)y$. □

For a domain A, $f \in S = A[\underline{t};\sigma,\underline{\delta}]$, and $\underline{a} \in A^n$, we define

$$V(f) = \{\underline{a} \in A^n \mid f(\underline{a}) = 0\} \text{ and}$$

The next proposition will put some structure on the set of roots of a polynomial $f \in S = A[\underline{t};\sigma,\underline{\delta}]$. The set $V(f)$ is naturally divided into conjugacy classes. For any $\underline{a} \in V(f)$, we consider the set

$$A_{\underline{a}} := \{x \in A \mid \exists \underline{b} \in A^n \text{ with } \underline{b}x = \sigma(x)\underline{a} + \underline{\delta}(x)\}$$

Since A is a domain we notice that if $x \in A_{\underline{a}}$ there exists a *unique* $\underline{b} \in A^n$ such that $\underline{b}x = \sigma(x)\underline{a} + \underline{\delta}(x)$. We will denote this unique $\underline{b}$ by $\underline{a}^x$. We put

$$E(f,\underline{a}) := \{x \in A_{\underline{a}} \mid f(\underline{a}^x) = 0\}$$

We recall that $\Delta(\underline{a}) = \{\underline{b} \in A^n \mid \underline{a} \sim \underline{b}\} = \{\underline{a}^x \mid x \in A\}$.

Proposition 5.5. *Let A be a domain, $\underline{a} \in A^n$, and $f(\underline{t}) \in S = A[\underline{t};\sigma,\underline{\delta}]$. Then*

1. *If $0 \neq x \in A$ is such that $\underline{b}x = \sigma(x)\underline{a} + \underline{\delta}(x)$ then $x \in \ker f(T_{\underline{a}})$ if and only if $f(\underline{b}) = 0$*
2. *$E(f,\underline{a}) = \ker f(T_{\underline{a}}) \cap A_{\underline{a}}$*
3. *$\ker f(T_{\underline{a}})$ and $E(f,\underline{a})$ are right $C^{\sigma,\delta}(\underline{a})$ modules.*
4. *$\Delta(\underline{a}) \cap V(f) = \{\underline{a}^x \mid x \in E(f,\underline{a})\} = \underline{a}^{E(f,\underline{a})}$.*
5. *Let $\Gamma = \{\underline{a} \in A^n \mid V(f) \cap \Delta(\underline{a}) \neq \emptyset\}$. Then $V(f) = \bigcup_{\underline{a}\in\Gamma}(\underline{a}^{E(f,\underline{a})})$.*

Proof. 1. The fact that $x \in \ker f(T_{\underline{a}})$ implies $f(\underline{b}) = 0$ is given in Proposition (4.5). Conversely if $f(\underline{b}) = f(\underline{a}^x) = 0$, we have $0 = f(\underline{a}^x)x = fx(\underline{a}) = f(T_{\underline{a}})(x)$.

2. This is clear from 1; above.

3. From Proposition (5.4), it is clear that $\ker f(T_{\underline{a}})$ is C-linear. Now let $x \in A_{\underline{a}}$ and $\underline{b} \in A^n$ be such that $\underline{b}x = \sigma(x)\underline{a} + \underline{\delta}(x)$. Let also $c \in C^{\sigma,\delta}(\underline{a})$, then $\sigma(c)\underline{a} + \delta(c) = \underline{a}c$ and we have $\underline{b}xc = \sigma(x)\underline{a}c + \underline{\delta}(x)c = \sigma(x)(\sigma(c)\underline{a} + \underline{\delta}(c)) + \underline{\delta}(x)c = \sigma(xc)\underline{a} + \underline{\delta}(xc)$. This shows that $xc \in A_{\underline{a}}$ and hence $A_{\underline{a}}$ is a right $C^{\sigma,\delta}(\underline{a})$ module. This yields the proof.

4. and 5. are clear. □

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